Math 3093 Jan 18, 2024

Review.

- A function f defined on $[-\pi, \pi]$ with $f(\pi) = f(-\pi)$ can be extended to a 2π-periodic function on $\mathbb{R}$. It is said to be a function on the circle.

- (Uniqueness Thm) Let f be an integrable function on the circle such that

$$\hat{f}(n) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Then $f(x_0) = 0$ if f is cts at x_0.

- (A convergence Thm)

Assume that f is continuous on the circle, and that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Then $S_N f(x) \rightrightarrows f(x)$ on the circle as $N \to +\infty$.

Next we give a sufficient condition on f such that

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty .$$

<u>Thm 1</u>. Let f be a twice cts differentiable on the circle. Then

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty .$$

To prove the above result, we first give a Lemma.

Lem 2. (1) If f is integrable on $[-\pi, \pi]$, then

$$|\hat{f}(n)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f| \, dx , \quad \forall \ n \in \mathbb{Z}.$$

(2) If f is C^1 on the circle, then

$$\widehat{f'}(n) = in \, \hat{f}(n) , \quad n \in \mathbb{Z}$$

Pf. (1) is clear. To see (ii), for $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} f'(x) e^{-inx} dx = f(x) e^{-inx} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (-in) f(x) e^{-inx} dx$$

$$= 0 + (in) \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Hence $\widehat{f'}(n) = in \, \hat{f}(n)$. ▨

Pf of Thm 2. Since f is C^2 on the circle,

$$\widehat{f''}(n) = (in)\,\widehat{f'}(n) = (in)^2\,\hat{f}(n), \quad n \in \mathbb{Z}.$$

So for $n \neq 0$

$$\hat{f}(n) = -\frac{\widehat{f''}(n)}{n^2}.$$

Notice that f'' is continuous on the circle, so $\exists\, M > 0$ s.t $|\widehat{f''}(n)| \le M$ for all $n \in \mathbb{Z}$.

It follows that

$$\hat{f}(n) = O\left(\frac{1}{n^2}\right), \quad n \in \mathbb{Z} \setminus \{0\}.$$

Thus $\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$. $\boxtimes$.

2.4. Convolutions.

Let f, g be two integrable functions on the circle.

Define their convolution $f*g$ on $[-\pi,\pi]$ by

$$f*g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy.$$

- For given x, $f(x-y)\, g(y)$ is integrable (in y) on the circle.

- $$\begin{aligned} S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(n)\, e^{inx} \\ &= \sum_{n=-N}^{N} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(y)\, e^{-iny}\, dy\right)\cdot e^{inx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} e^{in(x-y)}\right)\cdot f(y)\, dy \\ &= D_N * f(x) \end{aligned}$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ which is called the N-th Dirichlet kernel.

Remark: This is a motivation to study convolutions of functions.

A direct calculation show that

- $$D_N(x) = \begin{cases} \dfrac{\sin\left(N+\frac{1}{2}\right)x}{\sin\left(\frac{1}{2}x\right)} & \text{if } x \neq 0 \\ 2N+1 & \text{if } x = 0. \end{cases}$$

Below we give some basic properties of convolutions.

Prop. 4. Suppose that f, g, h are integrable on the circle.

Then

(1) $f*(g+h) = f*g + f*h$

(2) $(cf)*g = c(f*g) = f*(cg)$ for any $c \in \mathbb{C}$

(3) $f*g = g*f$.

(4) $(f*g)*h = f*(g*h)$

(5) $f*g$ is cts

(6) $\widehat{f*g}(n) = \hat{f}(n)\,\hat{g}(n)$.

Pf: We only prove ③, ⑤ and ⑥, and leave the others to you as an exercise.

③ $f*g = g*f$.

Recall that for a fixed x,

$$f*g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

$$\overset{\text{Letting } z=x-y}{=\!=\!=\!=} \frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-z)\, g(z)\, (-1)\, dz$$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-z)\, g(z)\, dz$$

(since $f(x-z)\,g(z)$ is 2π-periodic in z)

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-z)\, g(z)\, dz$$

$$= g*f(x).$$

(6) $\widehat{f*g}(n) = \hat{f}(n)\, \hat{g}(n)$.

Notice that

$$\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f*g(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy \right) e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \cdot g(x-y) e^{-in(x-y)}\, dy\, dx$$

(Fubini Thm: Let $F(x,y)$ be a Riemann integrable function on $[a,b] \times [c,d]$. Then

$$\int_a^b \int_c^d F(x,y)\, dy\, dx = \int_c^d \int_a^b F(x,y)\, dx\, dy$$

$$= \iint_{[a,b]\times[c,d]} F(x,y)\, dx\, dy$$
)

(by Fubini)
$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \underbrace{\int_{-\pi}^{\pi} g(x-y) e^{-in(x-y)}\, dx}\, f(y) e^{-iny}\, dy$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(n)\, f(y) e^{-iny}\, dy$$

$$= \hat{g}(n)\, \hat{f}(n).$$

(5) $f * g$ is cts on the circle.

We first prove the result in the case when g is cts on the circle.

Notice that g is uniformly cts on the circle.

Hence $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|g(y_1) - g(y_2)| < \varepsilon \quad \text{if} \quad |y_1 - y_2| < \delta.$$

Now for x_1, x_2 on the circle with $|x_1 - x_2| < \delta$,

$$
\begin{aligned}
& f * g(x_1) - f * g(x_2) \\
& = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x_1 - y)\, dy \\
& \quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, g(x_2 - y)\, dy \\
& = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \cdot \left(g(x_1 - y) - g(x_2 - y) \right) dy
\end{aligned}
$$

Hence

$$|f*g(x_1) - f*g(x_2)|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot |g(x_1-y) - g(x_2-y)|\, dy$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)| \cdot \varepsilon\, dy$$

$$= \varepsilon \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\, dy$$

$$\leq \varepsilon \cdot B, \quad \text{where } B = \sup_{y\in[-\pi,\pi]} |f(y)|.$$

Hence $f*g$ is uniformly cts on the circle.

Next we consider the general case when g is merely integrable on the circle.

We need an auxiliary result.

<u>Lem 2</u>: Let g be integrable on $[-\pi, \pi]$. Then for any $\varepsilon > 0$, $\exists$ a cts function h

on the circle such that

$$|h(x)| \leq \sup_{y \in [-\pi, \pi]} |g(y)| , \quad \forall \ x \in [-\pi, \pi]$$

and

$$\int_{-\pi}^{\pi} | g(x) - h(x) | \, dx < \varepsilon .$$

We postpone the proof of the above lemma a while.

Now we use it to prove the continuity of $f * g$.

By Lemma 2, we can take a sequence of cts functions (h_n) on the circle such that

$$\int_{-\pi}^{\pi} |g(x) - h_n(x)| \, dx \to 0 \quad \text{as } n \to \infty .$$

Now let us estimate

$$f*g(x) - f*h_n(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot h_n(x-y)\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\cdot\big(g(x-y) - h_n(x-y)\big)\, dy.$$

Hence

$$|f*g(x) - f*h_n(x)|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(y)|\cdot|g(x-y) - h_n(x-y)|\, dy$$

$$\leq \frac{B}{2\pi}\int_{-\pi}^{\pi} |g(x-y) - h_n(x-y)|\, dy$$

$$\left(\text{where } B := \sup_{y\in[-\pi,\pi]} |f(y)|\right)$$

$$= \frac{B}{2\pi}\int_{-\pi}^{\pi} |g(y) - h_n(y)|\, dy$$

It implies that

$$f * h_n(x) \rightrightarrows f * g(x) \quad \text{on the circle}$$

as $n \to \infty$.

Since $f * h_n$ is cts on the circle, it follows that $f * g$ is cts on the circle.

$\blacksquare$